APPROXIMATION THEORY
LECTURE 3

§3.1 The Fejér sums + Weierstrass

Def: (Fourier sums). For $f \in L_1(\mathbb{T})$, we define the Fourier sums
$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx).$$
$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \begin{bmatrix} \cos kt \\ \sin kt \end{bmatrix} dt.$$
Warning, not true that $\|S_n(f) - f\|_\infty \to 0 \ \forall f.$

Lemma: We have:
$$S_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} f(t) D_n(x-t)\,dt.$$
Dirichlet kernel.

Proof:
$$a_k \cos kx + b_k \sin kx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\cos(kt)\cos(kx) + \sin(kt)\sin(kx)\right] dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos(k(x-t))\,dt$$
Hence $D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx$
$$= \frac{\sin(n+1/2)x}{2\sin\left(\frac{x}{2}\right)}.$$

Def: Fejér sums. For $f \in L_1(\mathbb{T})$, we define the Fejér sums:
$$\sigma_n(f,x) = \frac{1}{n}\sum_{k=0}^{n-1} S_k(f,x).$$

Lemma: $\sigma_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F_n(x-t) f(t)\,dt,$
$$F_n(x) = \frac{1}{n}\frac{\sin^2 \frac{nx}{2}}{2\sin^2\frac{x}{2}} \geq 0.$$

LEMMA: $\sigma_n(f,x)$ is a positive linear operator (which S_n is not), and
$$\sigma_n \underbrace{\begin{bmatrix} \cos kx \\ \sin kx \end{bmatrix}} = \left(1 - \frac{k}{n}\right)\begin{bmatrix} \cos kx \\ \sin kx \end{bmatrix}$$
appears $(n-k)$ times in S_n, $n \geq k$.

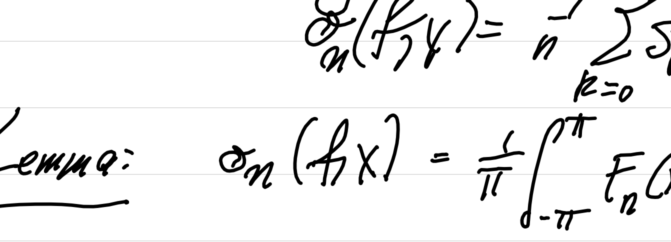
Hence, $\sigma_n \begin{bmatrix} \cos(kx) \\ \sin(kx) \end{bmatrix} \to \begin{bmatrix} \cos kx \\ \sin kx \end{bmatrix} \ \forall k, n \to \infty.$

Thm. (Weierstrass): the set $\mathcal{T} = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n$ is dense in $C(\mathbb{T})$.

Proof: Applying Korovkin to $\rho(x,t) = 1 - \cos(x-t) \geq 0$. yields $\sigma_n(f) \to f \ \forall f \in C(\mathbb{T})$.

§ 3.2 The Chebyshev alternation theorem:

Thm: The pol. $p^* \in P_n$ is the best approx. to $f \in C[a,b]$ iff $\exists (t_k)_{k=1}^{n+2}$ s.t. $f(t_k) - p^*(t_k) = (-1)^k \gamma$, $\gamma = \pm\|f-p^*\|_\infty$



Proof: (Sketch) suppose $\exists$ less than $n+2$ such points: can augment the points by adding roots of $f-p^*$ between t_k and add a polynomial ε s.t. $\varepsilon(\text{roots} < 0)$ and: $\|f - p - \varepsilon\| < \|f - p^*\|$.

Thm: The pol. $p^* \in \mathcal{T}_n$ is the best approx to $f \in C(\mathbb{T})$ iff $\exists (t_k)_{k=1}^{2n+2}$ s.t. $f(t_k) - p(t_k) = (-1)^k \gamma$, $\gamma = \pm\|f - p_*\|$

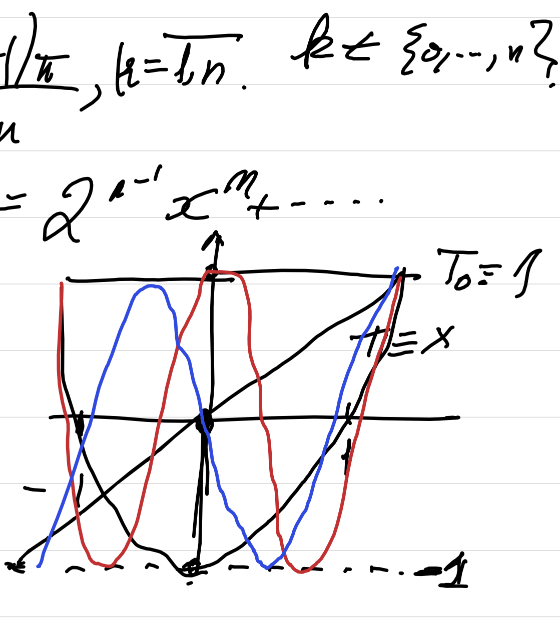
Remark: a) $n+2 = \dim P_n + 1$
b) $2n+2 = \dim \mathcal{T}_n + 1$.

Def: $T_n(x) = \cos(n \arccos x)$, $x \in [-1,1]$.
$T_n(\theta) = \cos n\theta$, $x = \cos\theta$, $\theta \in [0,\pi]$.

1) $\|T_n\| = 1$, $T_n(t_n^*) = (-1)^k$, $t_n = \cos\frac{\pi k}{n}$, $k \in \{0,\dots,n\}$

2) $T_n(x_n) = 0$, $x_k = \frac{(2k-1)\pi}{2n}$, $k = \overline{1,n}$.

3) $T_n \in P_n$, $T_n(x) = 2^{n-1}x^n + \dots$



APPROXIMATION THEORY

LECTURE 5

Thm (Jackson):

$$E_n(f) \leqslant C\cdot \omega(f, 1/n). \qquad f\in C(\mathbb{T})$$
$$\leqslant \frac{C}{n}\|f'\|_\infty \quad \text{if } f\in C^1.$$

Thm (2nd Jackson):

$$E_n(f) \leqslant \frac{C}{n^r} E_n(f^{(r)}) \leqslant \frac{C}{n^r}\,\omega(f^{(r)}, 1/n)$$

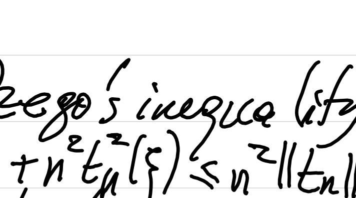
(Proof not needed / not examinable).

§ 5. Inverse theorems

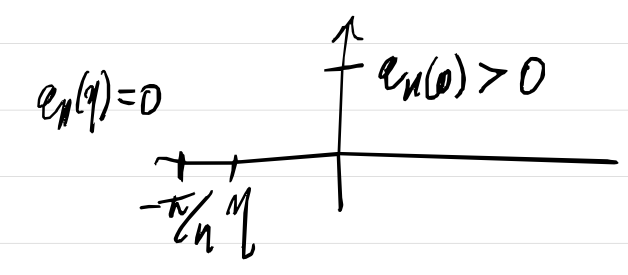
Thm (Bernstein): $\forall t_n \in \mathcal{T}_n$ on $\mathbb{T} = [-\pi,\pi)$, (trig polys.) we have $\|t_n'\|_\infty \leqslant n\cdot\|t_n\|_\infty$. The constant n is sharp. (Take $t_n^*(x) = \sin(nx)$).

Remark: Much more difficult for algebraic polynomials.

Proof: Will prove stronger Szegő's inequality:
$\forall \xi\in[-\pi,\pi)$, $[t_n'(\xi)]^2 + n^2 t_n^2(\xi) \leqslant n^2\|t_n\|_\infty$.
It is sufficient to prove it for **any** fixed particular ξ.



Suppose $t_n(\xi) = \alpha$, $t_n'(\xi) > 0$.
Consider $s_n(x) = \gamma\cos(nx)$, $\gamma > \|t_n\|_\infty$.
s_n is strictly increasing on $[-\pi/n, 0]$ and $\exists!\,\eta$ s.t. $s_n(\eta) = \alpha$. Take $\xi = \eta$.
On $t_k = \left(\frac{\pi k}{n}\right)_{k=-n}^{n-1}$ s_n alternates in sign, hence $e_n := s_n - t_n$ also alternates in sign (has the same sign as s_n on t_k), with $2n$ points of alternation.
$\Rightarrow$ e_n has $2n$ zeros at least (by periodicity), and exactly 1 zero in each (t_k, t_{k+1}).
Now, look $(-\pi/n, 0)$:



If $e_n'(\eta) \leqslant 0 \Rightarrow \#$.
$e_n'(\eta) = s_n'(\eta) - t_n'(\eta) > 0$
$\Rightarrow 0 < t_n'(\eta) < s_n'(\eta)$
$t_n(\xi) = s_n(\eta) \Rightarrow |t_n'(\xi)| < |s_n'(\eta)|$

Finish: $0 < t_n'(\eta) < s_n'(\eta) = \frac{n\gamma\sin n\eta}{} = n\sqrt{\gamma^2 - \gamma^2\cos^2 n\eta}$
$= n\sqrt{\gamma^2 - t_n(\eta)^2} \Rightarrow t_n'(\eta)^2 + n^2 t_n(\eta)^2 \leqslant \gamma^2 n^2$.
$\gamma > \|t_n\|$ (γ arbitrary) and we obtain the inequality if we let $\gamma \searrow \|t_n\|$. $\square$

Corollary: $\|t_n^{(k)}\|_\infty \leqslant n^k\cdot\|t_n\|_\infty$ (and is sharp).

§ 5.2 Inverse theorems:

Thm. (The first inverse theorem). We have
$$\omega(f, 1/n) \leqslant \frac{C}{n}\sum_{k=0}^{n} E_k(f).$$

Proof: Let t_n be best approximation poly $\in \mathcal{T}_n$ to f, $E_k = E_k(f)$. We have
$$\omega(f,\delta) \leqslant \omega(f - t_{2^m}, \delta) + \omega(t_{2^m}, \delta)$$
$$\leqslant 2E_{2^m} + \delta\cdot\|t_{2^m}'\|$$

$$E_{2^m} = \frac{1}{2^m}\sum_{k=1}^{2^m} E_{2^m} < \frac{1}{2^m}\sum_{k=1}^{2^m} E_k \quad \text{(monotonicity of } (E_k)_k)$$

$$\|t_{2^m}'\|_\infty = \|t_{2^m}' - t_0'\|_\infty \leqslant \|t_1' - t_0'\| + \sum_{s=1}^{m}\|t_{2^{s+1}}' - t_{2^s}'\|_\infty$$

$$\leqslant 2\cdot 2\cdot E_0 + \sum 2^{s+1}\, 2E_{2^s}$$

Bernstein, ($\|t_{m+k}' - t_k'\| \leqslant 2E_n(f)$).

$$\leqslant 4E_0 + 8\sum_{s=1}^{m-1} 2^{s-1}E_{2^s} \leqslant 4E_0 + 8\sum_{k=2}^{2^m-1} E_k$$
$$\leqslant 8\sum_{k=0}^{2^m-1} E_k.$$

Then, we derive
$$\omega(f,\delta) \leqslant 8(\delta + 2^{-m})\cdot\sum_{k=0}^{2^m} E_k(f).$$

With $\delta = 1/n$ & m s.t. $2^m \leqslant n < 2^{m+1}$
$$\text{RHS} \leqslant 8\left(\frac{1}{n} + \frac{2}{n}\right)\sum_{k=0}^{n} E_k(f).$$
$\square$

Def$^{\underline{n}}$ $f\in \mathrm{Lip}\,\alpha$, $\alpha\in(0,1]$ if $|f(x) - f(y)| \leqslant M\cdot|x-y|^\alpha$. $\forall x,y$

Thm: Given $f\in C(\mathbb{T})$, we have
$E_n(f) = \mathcal{O}(n^{-\alpha})$, $\alpha\in(0,1)$,
$\Leftrightarrow \omega(f,t) = \mathcal{O}(t^\alpha) \Leftrightarrow f\in \mathrm{Lip}\,\alpha$.

Proof: 1) By Jackson, $E_n(f) \leqslant c\cdot\omega(f, \frac{1}{n})$
$\underset{\mathrm{Lip}\,\alpha}{\leqslant} c'\cdot\frac{1}{n^\alpha}$

2) If $E_n(f) \leqslant \frac{c}{n^\alpha} \Rightarrow$ by inverse thm

$$\omega(f, 1/n) \leqslant \underbrace{\frac{c}{n}\cdot\sum_{k=0}^{n}\frac{1}{k^\alpha}}_{\text{Integral bound}} \leqslant \frac{c'}{n^\alpha}.$$

What if $E_n(f) = \mathcal{O}(\frac{1}{n})$?

Direct & inverse do **not** match.

1) $E_n(f) = \mathcal{O}(1/n)$
2) $\omega(f, 1/n) = \mathcal{O}(1/n)$.

$$\omega(f, 1/n) \leqslant \frac{c}{n}\cdot\sum_{k=1}^{n}\frac{1}{k} = \mathcal{O}\left(\frac{\log n}{n}\right).$$

Finally, Thm (Zygmund)

$$E_n(f) = \mathcal{O}(1/n) \Longleftrightarrow \omega_2(f, \tfrac{1}{n}) = \mathcal{O}(1/n).$$

Pf: $\Leftarrow$ by Jackson (Ex. 4.4).

$$\Rightarrow \omega_2(f, 1/n) \leqslant \frac{c}{n^2}\sum_{k=0}^{n} k\cdot E_k(f).$$

$\square$

APPROXIMATION THEORY
LECTURE 6

Examples class: Thursday 2 Nov MR 3 2pm

$\downarrow$ 5 lectures.

Hand in $\rightarrow$ pigeonhole at CMS Mo 30th October by 1pm (A. SHADRIN).

Existence and uniqueness of best approximation:

1) If $\mathcal{U} = \mathcal{U}_n$ ($\dim \mathcal{U} = n$) $\Rightarrow$ b.a. exists.
2) If $\mathcal{U}_n = \mathcal{P}_n, \mathcal{T}_n$, $\|\cdot\| = \|\cdot\|_\infty$ (max-norm) $\Rightarrow$ b.a. unique.

$$d_n(K)_\infty := \inf_{\mathcal{U}_n : \dim \mathcal{U}_n = n} \sup_{f \in K} \inf_{u \in \mathcal{U}_n} \|f-u\|_\infty$$

Kolmogorov n-width.

Def: $k, N \in \mathbb{N}$. $\Delta_* = \{a = \tau_0 < \underbrace{\tau_1 < \dots < \tau_N}_{N \text{ interior knots}} < \tau_{N+1} = b\}$.

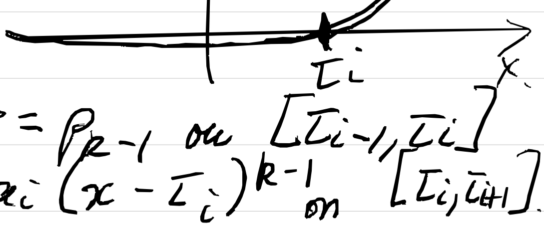
$$s \in S_k(\Delta_*) \iff \begin{cases} 1)\ s \in \mathcal{P}_{k-1}[\tau_i, \tau_{i+1}]\ \forall i. \\ 2)\ s \in C^{k-2}(\tau_{i-1}, \tau_{i+1}). \end{cases}$$

Generally, can relax C^{k-2} to C^{k-1-m} (deficiency m).

$$\mathcal{P}_{k-1} \subset S_{k,0}(\Delta_*) \subset \dots \subset S_{k,k}(\Delta_*) = \mathcal{P}_k(\Delta_*)$$

piecewise poly of deg $k-1$ (no smoothness).

Example: the truncated power
$$(x-\tau_i)_+^{k-1} := [\max\{0, x-\tau_i\}]^{k-1}$$



From $s \in C^{k-2} \Rightarrow$ If $s = p_{k-1}$ on $[\tau_{i-1}, \tau_i]$ then $s = p(x) + a_i (x-\tau_i)^{k-1}$ on $[\tau_i, \tau_{i+1}]$.

Hence,

Thm: Any $s \in S_k(\Delta_*)$ has a representation
$$s(x) = p_{k-1}(x) + \sum_{i=1}^{N} a_i (x-\tau_i)_+^{k-1}$$

Good bases: Truncated powers are not good $\rightarrow$ large support. $\hookrightarrow$ τ_i, τ_{i+1} almost lin. indep.

$$\dim S_k(\Delta_*) = k + N$$

Let $\mathcal{U}_n = (\varphi_j)_{j=1}^n$ and f is interpolated at x_i. Interpolant $u = \sum_{j=1}^n a_j \varphi_j$

$$\Rightarrow \sum_{j=1}^n a_j \varphi_j(x_i) = \underset{\overset{\shortparallel}{b_i}}{f(x_i)}, \quad i = 1, \dots, n.$$

$Aa = b$, $A = (\varphi_j(x_i))$, $b = (f(x)_i)$,
A is called the collocation matrix.

1) A is easily invertible (system easily solved). Say A is a band matrix is certainly an advantage.
2) $\kappa := \|A\| \|A^{-1}\| \leq$ constant (uniformly in n). $\kappa \geq 1$ always (small changes in RHS $b \rightarrow$ small changes in a).
3) Stability of (φ_j): $\sum_{j=1}^n a_j \varphi_j = u$.

B-spline: $M_i(t) = k[t_i, \dots, t_{i+k}](\cdot - t)_+^{k-1}$

Def: Div. difference of order k of $f \in C[a,b]$ on $(t_0, \dots, t_k)$ is the lead coeff-t of $p_k \in \mathcal{P}_k$ s.t. $p_k(t_i) = f(t_i)$, labelled
$f[t_0] = f(t_0)$,
$f[t_0, \dots, t_k] = 0$, $f \in \mathcal{P}_{k-1}$ and
$[t_0, \dots, t_k]\, x^k = 1$.

Properties: 1) Explicit form: $f[t_0, \dots, t_k] = \sum_{i=0}^{k} \frac{f(t_i)}{\omega'(t_i)}$
$\omega(x) = \prod (x - t_i)$.

$$p(x) = \sum_{v=0}^{k} \frac{f(t_v)}{\omega'(t_v)} \frac{\omega(x)}{(x-t_v)} \quad \text{lagrange poly.}$$

2) **Recurrence relation:**

$$f[t_0, \dots, t_k] = \frac{f[t_1 \dots t_k] - f[t_0, \dots, t_{k-1}]}{t_k - t_0}$$

3) Leibnitz formula: If $h = f \cdot g$.
$$h[t_0 \dots t_k] = \sum_{i=0}^{k} f[t_0 \dots t_i] \cdot g[t_i \dots t_k].$$

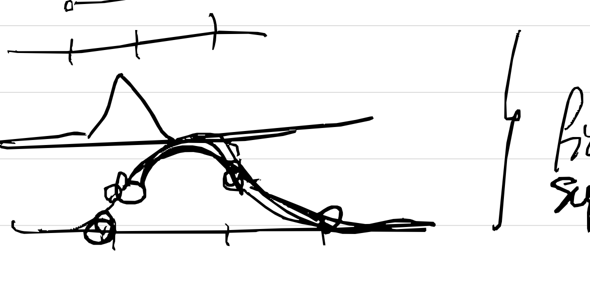
$$f[t_0, \dots, t_0] = f^{(k)}(t_0) \frac{1}{k!}$$

$$h^{(k)}(t_0) = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(t_0)\, g^{(k-i)}(t_0).$$

$k = 1$

$k = 2$

$k = 3$



finite support.

APPROXIMATION THEORY

LECTURE 7

§7. B-splines (contd)

Def: Given $k, n \in \mathbb{N}$, and $\Delta = (t_i)_{i=1}^{n} \subset [a,b]$

$\Delta = \{a \le t_1 \le t_2 \le \dots \le t_{n+k} \le b\}$

$M_i(t) = k\,[t_i, \dots, t_{i+k}](\cdot - t)_+^{k-1}$ ← normalised $\int_{t_i}^{t_{i+k}} M_i(t)\,dt = 1$

$N_i(t) = (t_{i+k} - t_i) \times [t_i, \dots, t_{i+k}](\cdot - t)_+^{k-1}$ $(\sum N_i = 1)$

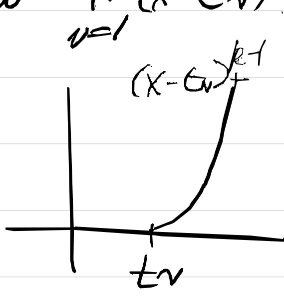
$$M_i(t) = \frac{k}{t_{i+k} - t_i} N_i(t).$$

Properties: 1) Spline structure

$(M_i) \in S_k(\Delta)$

Proof: If (t_i) are distinct, apply formula for divided difference:

$$f[t_i, \dots, t_{i+k}] = \sum_{\nu=i}^{i+k} \frac{f(t_\nu)}{w'(t_\nu)}, \quad w = \prod_{\nu=1}^{i+k} (x - t_\nu)$$



$$M_i(t) = k \sum_{\nu=1}^{i+k} \frac{(t_\nu - t)_+^{k-1}}{w'(t_\nu)}$$

$$= (-1)^k k \sum_{\nu=1}^{i+k} \frac{(t - t_\nu)_+^{k-1}}{w'(t_\nu)}$$

and $(-1)^{k-1}(t_\nu - t)_+^{k-1} + (t - t_\nu)_+^{k-1} = \underbrace{(t - t_\nu)^{k-1}}_{g}$

$g[\dots] = 0.$

$S_k(\Delta) \subset C^{k-2}$, $S_{k,m}(\Delta)$

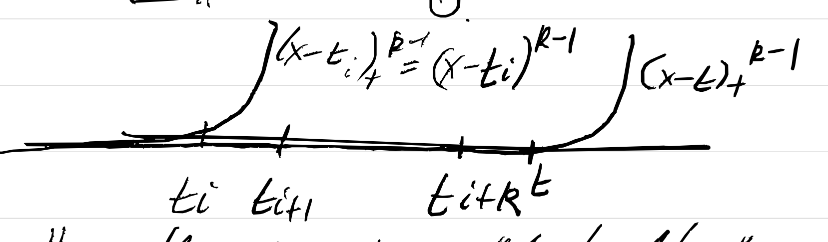
2) B-splines have finite support $\operatorname{supp} M_i = [t_i, t_{i+k}]$.

Def: $\operatorname{supp}(f) := \{f(x) \neq 0\}$

Proof: if $t > t_{i+k} \Rightarrow (x - t)_+^{k-1} = 0$ on $x = t_i, \dots, t_{i+k}$. $\Rightarrow [\dots](\cdot - t)_+^{k-1} = 0$.

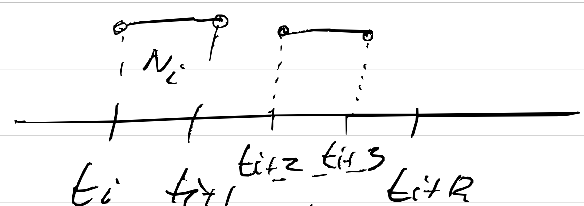
if $t < t_i \Rightarrow (x-t)_+^{k-1} = (x-t)^{k-1}$ on $x = t_i \dots t_{i+k}$

— " — $= 0$.

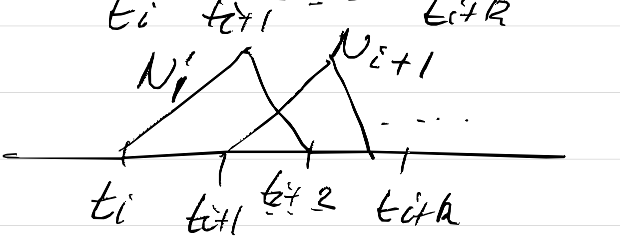


How the B-splines "look like"!

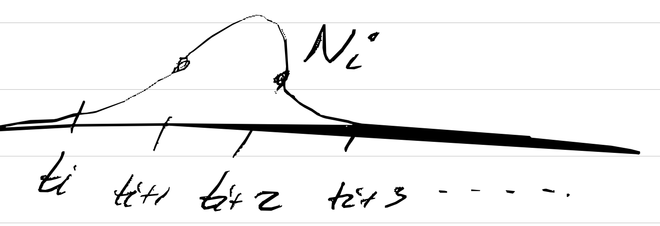
$k=1$



$k=2$:



$k=3$:



3) Peano kernel: we have

$N(f) = \int K(t) \cdot f(t)\,dt$ ← Peano kernel.

Take the Taylor formula for f with Cauchy remainder.

$f(x) = p_{k-1}(f, x) + \frac{1}{(k-1)!} \int_a^b (x - t)_+^{k-1} \cdot f^{(k)}(t)\,dt.$

$f \in C^k[a,b]$ at $x_0 = a$

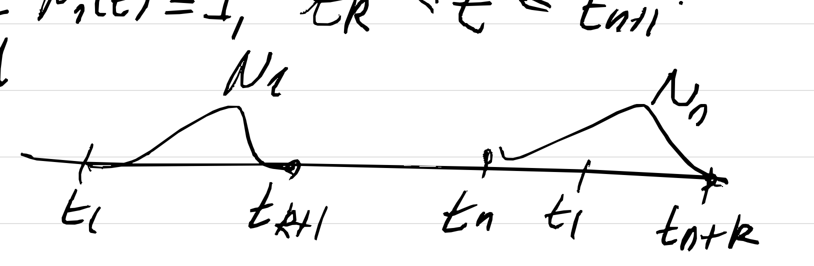
Apply divided differences to both sides. Apply div. diff. to both sides: (wrt x).

$$f[t_i, \dots, t_{i+k}] = \frac{1}{k!} \int_a^b M_i(t) f^{(k)}(t)\,dt$$

$$= \frac{1}{k!} \int_{t_i}^{t_{i+k}} M_i(t) f^{(k)}(t)\,dt.$$

4) Normalisation: a) $\int_{t_i}^{t_{i+k}} M_i(t)\,dt = 1.$

b) $\sum_{i=1}^{n} N_i(t) \equiv 1$, $t_k \le t \le t_{n+1}$:



Proof: a) Take $f(x) = x^k$ in $(*)$.

b) $N_i(t) = ([t_i, \dots, t_{i+k}] - [t_i, \dots, t_{i+k-1}])(\cdot - t)_+^{k-1}$

$\Rightarrow \sum_{i=1}^{n} N_i(t) = ([t_{n+1}, \dots, t_{n+k}] - [t_1, \dots, t_k])(\cdot - t)_+^{k-1}$

if $t \le t_{n+1} \Rightarrow [t_{n+1}, \dots, t_{n+k}](x - t)_+^{k-1}$
$= [\dots](x-t)^{k-1} = 1.$

if $t \ge t_k \Rightarrow [t_1, \dots, t_k](x-t)_+^{k-1} = 0$

5) Recurrence relation (de Boor)

$$\boxed{\tfrac{1}{k} M_{i,k}(t) = \frac{1}{(k-1)} \left[\frac{t - t_i}{t_{i+k} - t_i} \cdot M_{i,k-1}(t) + \frac{(t_{i+k} - t)}{(t_{i+k} - t_i)} M_{i+1,k-1}(t) \right]}$$

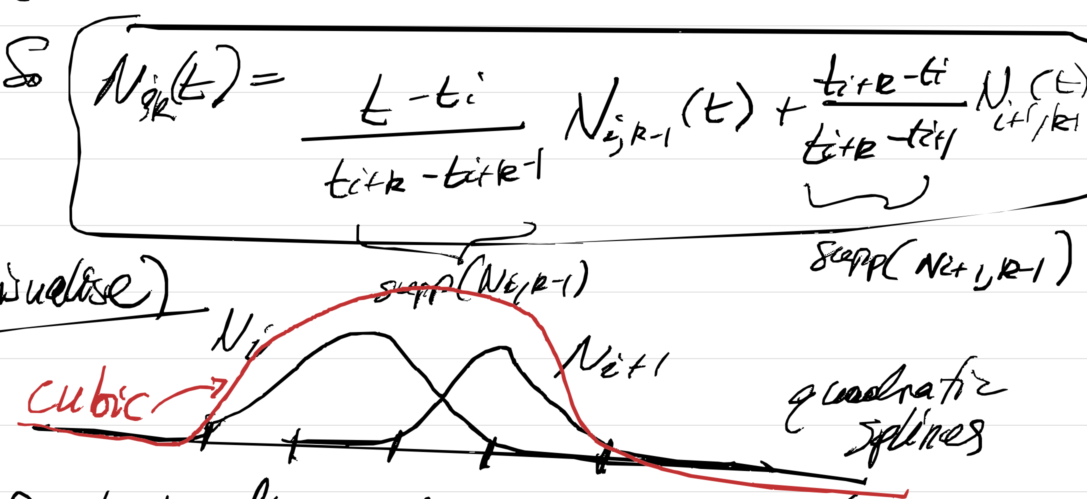
Proof: Use Leibniz rule for

$h(x) = (x - t)_+^{k-1} = (x - t)(x - t)_+^{k-2} = f(x) g(x)$

(see handout).

So $$\boxed{N_{i,k}(t) = \underbrace{\frac{t - t_i}{t_{i+k} - t_{i+k-1}}}_{\operatorname{supp}(N_{i,k-1})} N_{i,k-1}(t) + \underbrace{\frac{t_{i+k} - t_i}{t_{i+k} - t_{i+1}}}_{\operatorname{supp}(N_{i+1,k-1})} N_{i+1,k-1}(t)}$$

(Visualise)

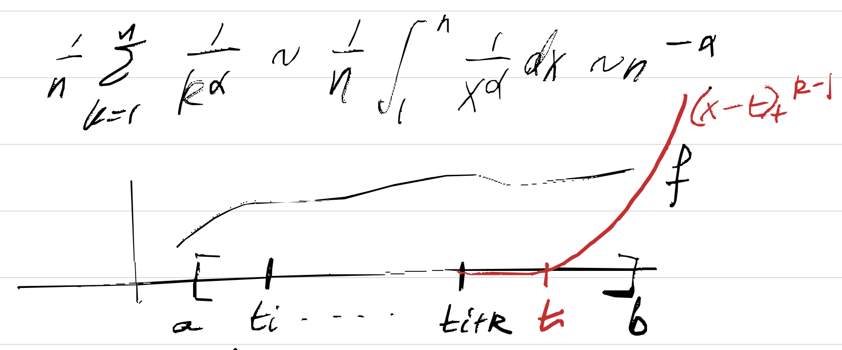


Positivity follows from recurrence relation and base case.

Approximation Theory
Lecture 8

1) $\frac{1}{n}\sum_{k=1}^{n} \frac{1}{k^\alpha} \sim \frac{1}{n}\int_1^n \frac{1}{x^\alpha}\,dx \sim n^{-\alpha}$

2)



$f \longmapsto f[t_i, t_{i+1}, \ldots, t_{i+k}]$
$= \sum \frac{f(t_i)}{\omega'(t_i)} \qquad \omega(x) = \prod (x - t_\nu)$.

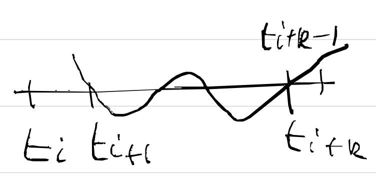
$f_t(x) = (x-t)_+^{k-1}$

$f_t[\cdots] = c(t) = N_i(t)$.

$\operatorname{supp} N_i(t) = [t_i, t_{i+k}]$

Def: $k, n \in \mathbb{N}$. $\Delta = (t_i)_{i=1}^{n+k}$

1) $w_i(x) = \prod_{\nu=i+1}^{i+k-1} (x - t_\nu)$



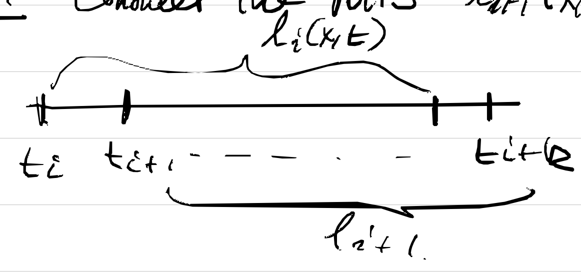
2) $\varphi_i(x) = \frac{1}{(k-1)!} w_i(x)$.

3) $l_i(x)$ interpolates $(x-t)_+^{k-1}$ on $t_i, \ldots, t_{i+k}$.

Lemma: (Lee's f-la). For any $\Delta = (t_i)_{i=1}^{n+k}$

$w_i(x) \cdot N_i(t) = l_{i+1}(x,t) - l_i(x,t)$ $\forall x, t, i = 1, \ldots, n$.

Proof: Consider the RHS $l_{i+1}(x,t) - l_i(x,t)$



$\Rightarrow l_{i+1}(x,t) - l_i(x,t) = c(t) \cdot w_i(x)$.

Then, $c(t) =$ (lead coeff. of l_{i+1}) $-$ (lead coeff. of l_i) $= [t_{i+1}, \ldots, t_{i+k}] - [t_i, \ldots, t_{i+k-1}]$
$= \left([t_{i+1}, \ldots, t_{i+k}] - [t_i, \ldots, t_{i+k-1}]\right)(x-t)_+^{k-1}$
$= N_i(t) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](x-t)_+^{k-1}$

Proposition (Marsden's identity):

$$(x-t)^{k-1} = \sum_{i=1}^{n} w_i(x) \cdot N_i(t), \ \forall x, t \in [t_k, t_{n+1}].$$

Proof: $\sum_{i=1}^{n} w_i(x) N_i(t) = l_{n+1}(x,t) - l_1(x,t)$.
(telescopic series).

$t > t_k \Rightarrow l_1(x,t) = \underbrace{(x-t)_+^{k-1}}_{=0}$ on $x = t_1, \ldots, t_k$.

$t < t_{n+1} \Rightarrow l_{n+1}(x,t) = (x-t)_+^{k-1}$ on $x = t_{n+1}, \ldots, t_{n+k}$
$= (x-t)^{k-1}$

$\Rightarrow \sum_{i=1}^{n} w_i(x) \cdot N_i(t) = (x-t)^{k-1}$

Corollary: $\mathcal{P}_{k-1}[t_k, t_{n+1}]$ belongs to $\operatorname{span}(N_i)_{i=1}^{n}$

Proof: $\frac{1}{(k-1)!}(x-t)^{k-1} = \sum \varphi_i(x) N_i(t)$

Diff $k-1-m$ times wrt x

$\frac{1}{m!}(t-x)^m = \sum (-1)^m \varphi_i^{(k-1-m)}(x) \cdot N_i(t)$

Take $x = a$, $\qquad a = t_1$

(*) $\frac{1}{m!}(t-a)^m = \sum (-1)^m \varphi_i^{(k-1-m)}(a) N_i(t)$

$m = 0, \ldots, k-1$.

Examples: 1) Take $m = 0 \Rightarrow$
$1 = \sum \underbrace{1 \cdot \varphi_i^{(k-1)}(x)}_{1} N_i(t)$
$\Rightarrow \sum N_i(t) \equiv 1$
$t \in [t_k, t_{n+1}]$.

2) Take $m = 1$, $(t-x) = \sum (-1) \underbrace{\varphi_i^{(k-2)}(x)}_{x - \frac{(t_{i+1} + \cdots + t_{i+k-1})}{k-1}} N_i(t)$

$\left[\varphi_i = \frac{1}{(k-1)!}\left(x^{k-1} - \left(\sum_{\nu=i+1}^{i+k-1} t_\nu\right) x^{k-2} + \text{l.o.t.}\right)\right]$.

$\Downarrow$

Now, $(t-x) = \sum (-1)(x - t_i^*) N_i(t)$

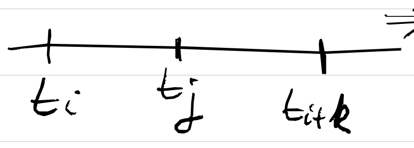
$\Rightarrow t = \sum_{i=1}^{n} t_i^* N_i(t)$

$\Rightarrow \underbrace{P_1(t)}_{a_1 t + b_1} = \sum_{i=1}^{n} p_1(t_i^*) \cdot N_i(t)$.

Corollary $\{(t-t_j)_+^{k-1}\}_{j=k}^{n+1} \subset \operatorname{span}\{N_i(t)\}_{i=1}^{n}$
for $t \in [t_k, t_{n+1}]$.

Proof: In (*), take $m = k-1$, and $x = t_j$,

$\frac{1}{(k-1)!}(t-t_j)^{k-1} = \sum_{i=1}^{n} (-1)^{k-1} \varphi_i(t_j) N_i(t)$



$\Rightarrow \varphi_i(t_j) = 0, t_i < t_j < t_{i+k}$
$\Longleftrightarrow t_j \in \{t_{i+1}, \ldots, t_{i+k-1}\}$.

$\ldots = \left(\sum_{i+k \leq j} + \sum_{i \geq j}\right)(-1)^{k-1} \varphi_i(t_j) N_i(t)$

Next, $t \geq t_j$, then the first sum

$\sum_{i+k \leq j} (-1)^{k-1} \varphi_i(t_j) N_i(t) = 0$ $\qquad \operatorname{supp} = [t_i, t_{i+k}]$

So we have

$t \geq t_j$: $\frac{1}{(k-1)!}(t-t_j)^{k-1} = \sum_{i \geq j} \smile N_i(t)$

$t \leq t_j$: $\Rightarrow \sum_{i \geq j} \text{---} N_i(t) = 0$.

Thm: B-splines form a basis $\mathcal{S}_k(\Delta)$ and $\mathcal{S}_k(\Delta_*)$.

Δ_*: $a = \tau_0 < \tau_1 < \cdots < \tau_N < \tau_{N+1} = b$

Δ: $a = t_k < t_{k+1} < \cdots < t_n < t_{n+1} = b$
$\leq t_{n+2} \leq \cdots \leq t_{n+k}$.

APPROXIMATION THEORY
LECTURE 9

§9 Dual Functionals:

LEMMA: Any $p \in \mathbb{P}_{n-1}$ has the following B-spline expansion

$$p(t) = \sum_{i=1}^{n} \lambda_i(p) N_i(t), \quad t \in [t_k, t_{n+1}]$$

where $\lambda_i(p) = \lambda_i(p, x) = \sum_{m=0}^{k-1} p^{(m)}(x) \cdot \psi_i^{(k-1-m)}(x), \forall x \in \mathbb{R}$.

$t_i \; t_{i+1} \; \cdots \; t_{i+k-1} \; t_{i+k}$

$$\psi_i(x) = \frac{1}{(k-1)!}(x - t_{i+1}) \times \cdots \times (x - t_{i+k-1})$$

Proof: Taylor expansion of $p(t)$ at $t = x$, + Marsden identity

$$p(t) = \sum_{m=0}^{k-1} p^{(m)}(x) \underbrace{\frac{(t-x)^m}{m!}}_{\text{Marsden}}$$

$$= \sum_{m=0}^{k-1} p^{(m)}(x) \cdot \underbrace{\sum_{i=1}^{n} (-1)^m \psi_i^{(k-1-m)}(x) \cdot N_i(t)}_{\text{Marsden}}.$$

$$= \sum_{i=1}^{n} \Bigg[\underbrace{\sum_{m=0}^{k-1} (-1)^m p^{(m)}(x) \psi_i^{(k-1-m)}(x)}_{\lambda_i(p,x)} \Bigg] \cdot N_i(t)$$

Def: the functional $\lambda_i(f, x) = \sum_{i=1}^{n} (-1)^m f^{(m)}(x) \psi_i^{(k-1-m)}(x)$ is called the ~~he~~ de Boor-Fix functional.

Examples: 1) $p \equiv 1 \Rightarrow \lambda_i(p, x) = 1 \cdot 1 = 1$.

$$\Rightarrow 1 \equiv \sum_{i=1}^{n} N_i(t), \quad t \in [t_k, t_{n+1}]$$

2) $p \in \mathbb{P}_1 \Rightarrow \lambda_i(p, x) = p(x) - p'(x)(x - t_i^*)$

$$t_i^* = \frac{t_{i+1} + \cdots + t_{i+k-1}}{k-1}.$$

$$\Rightarrow p(t) = \sum_{i=1}^{n} p(t_i^*) N_i(t).$$

LEMMA: for any $p \in \mathbb{P}_{k-1}$, we have that $\lambda_i(p, x) = \lambda_i(p)$, $\forall x \in \mathbb{R}$.
In particular $\lambda_i(p, \xi_i) = \lambda_i(p)$ $\forall \xi_i$.

Proof: B-splines (N_i) form a basis $\Rightarrow$ all expansion coeffs are uniquely determined

$$p(t) = \sum_{i=1}^{n} \underbrace{\lambda_i(p)}_{\text{uniquely determined}} \cdot N_i(t).$$

Thm: Any $s \in S_k(\Delta)$ has the following B-spline expansion

$$(*) \quad s(t) = \sum_{i=1}^{n} \overbrace{\lambda_i(s, \xi_i)}^{a_i} N_i(t).$$

$$\forall \xi_i \in [t_i, t_{i+k}].$$

In other words, (λ_i) are dual functionals to $(N_i)_{i=1}^{n}$, i.e. $\lambda_i(N_j, \xi_i) = \delta_{ij}$.

Proof: Let $\xi_i \in [t_\ell, t_{\ell+1}] \subset [t_i, t_{i+k}]$.

Let $p_j = N_j\big|_{[t_\ell, t_{\ell+1}]} \in \mathbb{P}_{k-1}$. Then, from $(*)$,

$$p_j = \sum \lambda_i(p_j, \xi_i) N_i(t)\Big|_{t_\ell, t_{\ell+1}} \text{ for } t \in [t_\ell, t_{\ell+1}]$$

$$= \sum_{i=\ell+1-k}^{\ell} \lambda_i(p_j, \xi_i) p_i$$

$t_{\ell+1-k} \quad t_\ell \quad t_{\ell+1}$

Now, B-splines reproduce $p \in \mathbb{P}_{k-1}[t_\ell, t_{\ell+1}]$
$\dim \mathbb{P}_{k-1} = k$ $\Rightarrow$ (p_j) are lin. independent on $[t_\ell, t_{\ell+1}]$

$$\Rightarrow \lambda_i(p_j) = \delta_{ij}$$

$$\Rightarrow \lambda_i(N_j, \xi_i).$$

$\square$

Conclusion: Not only dual functionals, but locally supported functionals.

Corollary: if $s \equiv 0$ on $[t_\ell, t_{\ell+1}]$ (but may not be zero elsewhere), then in $s = \sum_{i=1}^{n} a_i N_i$, we have $a_{\ell+1-k}, \ldots, a_\ell = 0$.

$t_\ell \quad t_{\ell+1}$

§9.2 Boundedness of λ_1.

We will prove that λ_i are bounded on $S_k(\Delta) \subset C[t_k, t_{n+1}]$.

LEMMA: $\forall s \in S_k(\Delta)$, we have

$$|\lambda_i(s, \xi_i)| \le d_k \|s\|_{C[t_i, t_{i+k}]}.$$

i.e., (λ_i) are uniformly bounded.

Proof: Let $[t_\ell, t_{\ell+1}]$ be a largest subinterval of $[t_i, t_{i+k}]$

$t_i \quad t_\ell \quad t_{\ell+1} \quad t_{i+k}$.

$$\Rightarrow \frac{t_{i+k} - t_i}{t_{\ell+1} - t_\ell} \le k.$$

Then, $|\lambda_i(s, \xi_i)| \le \sum_{m=0}^{k-1} |s^{(m)}(\xi_i)| \cdot |\underbrace{\psi_i^{(k-1-m)}(\xi_i)}|$

Markov inequality:

$$\|p_n^{(m)}\|_{[a,b]} \le \frac{C_{m,n}}{|b-a|^m} \cdot \|p\|_{C[a,b]}$$

Thus, $|s^{(m)}(x)| \le \dfrac{C_m'}{|t_{\ell+1} - t_\ell|^m} \|s\|_{C[t_\ell, t_{\ell+1}]}$

$\in \mathbb{P}_m$ with zeros in $[t_i, t_{i+k}]$.
$\le c_m |\xi_i - \eta_1| \cdots |\xi_i - \eta_m|$.
$\le c_m |t_{i+k} - t_i|^m$.

$$\Rightarrow |\lambda_i(s, \xi_i)| \le c_m \cdot C_m' \frac{|t_{i+k} - t_i|^m}{|t_{\ell+1} - t_\ell|^m} \|s\|_{[t_\ell, t_{\ell+1}]}$$

$$\le \underbrace{C_m \cdot C_m' \cdot k^m}_{d_k} \cdot \|s\|_{[t_i, t_{i+k}]}.$$

B-spline basis condition number

Approximation Theory
Lecture 10.

<u>Last time:</u> Given $S_k(\Delta)$, and the B-spline basis $(N_i)_{i=1}^{n}$, $\Delta = (t_i)_{i=1}^{n+k}$
$\exists (\lambda_i) : C[a,b] \to \mathbb{R}$ s.t. $\lambda_i(N_j) = \delta_{ij}$.
and $|\lambda_i(f)| \le \underline{d_k} \cdot \|f\|_{C[t_i, t_{i+k}]}$

d_k is independent of Δ!

<u>Remark:</u> By the Hahn-Banach theorem, any bdd lin. functional on a subspace of $U \subset X$ can be extended to the entire space X with the same norm:
$S(\Delta) \to C[a,b]$, t_k, t_{n+1}
So we can extend bounded λ_i from $S_k[a,b]$ to $C[a,b]$. $\lambda_i : C[a,b] \to \mathbb{R}$, and $|\lambda_i(f)| \le d_k \cdot \|f\|_{C[t_i, t_{i+k}]}$

<u>Markov's inequality:</u> On $[-1,1]$, we have
$\|p_m'\| \le m^2 \|p_m\|_{C[-1,1]}$, $p_m = T_m$ — equality.
dilation of $[a,b]$ depends on $[a,b]$.

$$\|p_m'\|_{[a,b]} \le \frac{C_m}{|b-a|} \|p_m\|_{C[a,b]}.$$

<u>Thm:</u> Any $s \in S_k(\Delta)$ has a representation
$$s(t) = \sum_{i=1}^{n} \underline{\lambda_i(s)} N_i(t).$$

where $\lambda_i(N_j) = \delta_{ij}$ and $\|\lambda_i(f)\| \le d_k \cdot \|f\|_{[t_i, t_{i+k}]}$.

<u>Def:</u> Quasi-interpolant of f is defined as
$$f \in C[a,b] \longmapsto Q(f) := \sum_{i=1}^{n} \lambda_i(f) \cdot N_i(t).$$

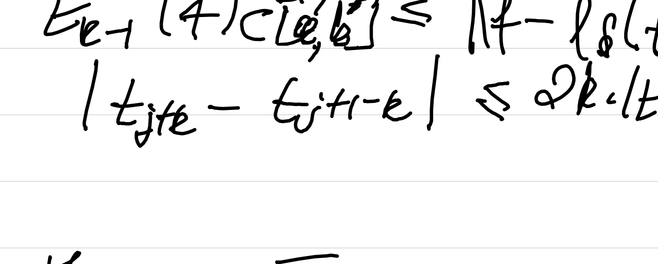
<u>Lemma:</u> $Q(f)$ is a linear projector onto $S_k(\Delta)$, and
$$\|Q(f)\|_{C[t_j, t_{j+1}]} \le d_k \cdot \|f\|_{C[t_{j+1-k}, t_{j+k}]}$$

<u>Proof.</u> Projector $\to$ clear.

<u>Estimate:</u> $t \in [t_j, t_{j+1}]$
$$|Q(f,t)| = \left|\sum_{i=1}^{n} \lambda_i(f) \cdot N_i(t)\right|$$
support
$$= \left|\sum_{i=j+1-k}^{j} \lambda_i(t) N_i(t)\right|, \quad \sum N_i \equiv 1$$
$$\le \max_{j+1-k \le i \le j} |\lambda_i(f)| = \|f\|_{C[t_{j+1-k}, t_{j+k}]}$$

<u>Thm:</u> With $I_j^* = [t_{j+1-k}, t_{j+k}]$, we have
$$E_{k,\Delta}(f) \le \|Q(f) - f\| \le C_k \cdot E_{k-1}(f)_{C[I_j^*]}$$
$\le c_k' \cdot |t|^k \cdot \|f^{(k)}\|_{C[I_j^*]}$
$|t| \to \max |t_{i+1} - t_i|$
$E_{k-1} \to$ polyn. approx
spline approx.
$|t| = 1/n$ for uniform partition.

<u>Proof:</u> Use Lebesgue inequality
$$\|f - U(f)\| \le (1 + \|U\|) \cdot \mathrm{dist}(f, U).$$
$\|f - Q(f)\|_{C[t_j, t_{j+1}]} = \|f - p - Q(f-p)\| \to$ a pol. of b.a. on I_j^*
$\le (1 + \|Q\|) \cdot \|f - p\|_{C(I_j^*)}$
$\|f - p\|_{C(I_j^*)} = E_{k-1}(f)_{C(I_j^*)}$.



$$E_{k-1}(f)_{C[a',b']} \le \|f - \ell_\delta(f)\| \le \frac{1}{2^{k-1}} \cdot \frac{1}{k!} \|f^{(k)}\|_{[a,b]}$$
$|t_{j+k} - t_{j+1-k}| \le 2k \cdot |t|$
ℓ_δ: interp. on $\delta \subset [a,b]$, $\ell_\delta \in P_{k-1}$. (Ex 3.5).

<u>Thm</u> $$E_{k,\Delta}(f) \le C_k \cdot |t|^k \cdot \|f^{(k)}\|.$$

<u>Inverse estimate</u> Does it follow from
$E_{k,\Delta}(f) = O(|t|^k) \Rightarrow f \in W_\infty^k$
$f \in C^k$.
Generally no, but if $E_{k,n}(f) = O\left(\frac{1}{n^k}\right)$ $\forall n$ then yes (uniform partition).
And if $o\left(\frac{1}{n^k}\right) \Rightarrow f \in P_{k-1}$.

§ 10.2. Spline interpolation

<u>Want:</u> Given $(x_i), (y_i)$, find <u>smooth</u> f s.t. $f(x_i) = y_i$, $i = 1, \dots, n$

1) Take $f \in U_n$, specifically $U_n = S_k(\Delta)$
$s(x_i) = y_i$.
$$s(x_i) = \sum_{j=1}^{n} c_j N_j(x_i) = y_i, \quad i = 1, \dots, n.$$
$$\iff A\bar{c} = \bar{y}, \quad A = (a_{ij}) = (N_j(x_i))_{i,j=1}^{n}.$$

<u>Theorem:</u> A is non-singular (invertible)
$\iff N_i(x_i) > 0 \iff x_i \in (t_i, t_{i+k})$.
$(x_i), (t_i)$ are strictly increasing.

$\Longrightarrow$ Ex 10.3.

(Next examples class, lectures 6–10)

APPROXIMATION THEORY
LECTURE 11

Def: Given $\Delta, \bar{x}$ satisfying Schoenberg–Whitney conditions, we get the spline interpolation liner projector $P_k : C[a,b] \to S_k(\Delta)$
$$a \to t_k, \quad b \to t_{n+k}$$
and then by the Lebesgue inequality:
$$E_{k,\Delta}(f) \le \|f - P_x f\| \le (1 + \|P_x\|) E_{k,\Delta}(f).$$

So we need a bound for $\|P_x\|$.

LEMMA: We have the following estimate:
$$\|P_x\|_\infty \le \|A_x^{-1}\|, \quad A_x = (N_j(x_i))_{i,j=1}^n$$

Proof: Let $P_x(f) = \sum_{j=1}^n a_j N_j \Rightarrow$
$$\|P_x\| \le \max |a_j| \underbrace{\sum N_j}_{\ge 0} \le \|a\|_\infty .$$

But $A_x \bar{a} = f(x_i) \Rightarrow \bar{a} = A_x^{-1} \cdot f(x_i)_{i=1}^n$
$$\Rightarrow \|\bar{a}\|_\infty = \|A_x^{-1} \cdot f|_x\|_\infty \le \|A_x^{-1}\| \cdot \|f\|_\infty.$$

$$\Rightarrow \|P_x(f)\|_{L_\infty} \le \|A_x^{-1}\|_{\ell_\infty} \cdot \|f\|_{L_\infty}$$

$$\Rightarrow \|P_x\|_{L_\infty} \le \|A_x^{-1}\|_{\ell_\infty}.$$

$X = \mathbb{R}^n$, $x \in \mathbb{R}^n$ introduce any norm $\|x\|_*$, e.g. $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ (Euclidean)

$\|x\|_\infty := \max |x_i|$

$\|x\|_1 = \sum |x_i|.$

Induced matrix norm $\|A\|_* = \sup_x \frac{\|Ax\|_*}{\|x\|_*}$.

CLAIM: $\|A\|_2 = \sqrt{\rho(A^T A)} = |\lambda_{max}(A)|$ if $A = A^T$

(spectral radius).

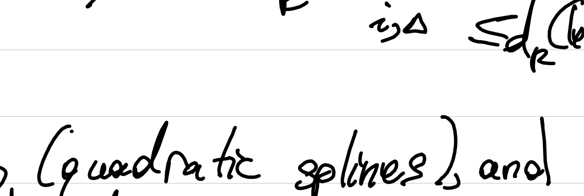
$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^n |a_{ij}| \leftarrow$ Ex 11.1

Remark: Q: Is $\|P_x\| \le \|A_x^{-1}\|$ any good?

A: Yes, (to a certain extent), because
$$\frac{1}{d_k} \|A_x^{-1}\| \le \|P_x\|, \text{ where } d_k = \max_{i, \Delta} \|\lambda_i\| \le k 2^k$$
$\le d_k$ (lecture 9)

Examples: a) $k = 3$, (quadratic splines), and equidistant delta and choose $x_i = i + 3/2$

Then $N_{i-1}(x_i) = \frac{1}{8}$
$N_i(x_i) = 6/8$
$N_{i+1}(x_i) = 1/8$
$\Big\} \Sigma = 1.$

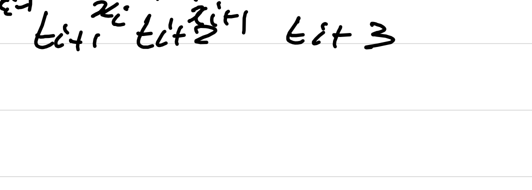


$$A_x = \begin{bmatrix} 6/8 & 1/8 & & 0 \\ 1/8 & 6/8 & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \end{bmatrix} \Rightarrow \|A_x^{-1}\| \le \left|\frac{6}{8} - \frac{2}{8}\right|^{-1} = 2.$$

$d_3 = 3. \Rightarrow \frac{2}{3} \le \|P_x\| \le 2$

b) Take $x_i = i + 2$

$$A_x = \frac{1}{2} \begin{bmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \end{bmatrix}$$



$$\Rightarrow A_x^{-1} = 2 \begin{bmatrix} 1 & -1 & 1 & \cdots \\ & 1 & -1 & \cdots \\ & & \ddots & \ddots \\ & & & 1 \end{bmatrix} \text{ alternating signs.}$$

$\Rightarrow \|A_x^{-1}\| = 2n$

$\Rightarrow \frac{2n}{3} \le \|P_x\| \le 2n$

Def^n: $A \in \mathbb{R}^{n \times n}$ is called $\overbrace{\text{diagonally dominant}}^{\text{strictly}}$ (SDD) (by rows if) $|a_{ii}| - \sum_{j \ne i} |a_{ij}| \ge \gamma > 0.$

LEMMA: If A is SDD with γ, then $\|A^{-1}\| \le \frac{1}{\gamma}$.

Defn: A is called totally positive (TP) if all its minors have non-negative determinant.

Thm (Karlin): $\forall \Delta, \forall x, \forall k$ A_x is TP

LEMMA: If A is invertible and TP, then

a) A^{-1} is checkerboard $\iff A^{-1}(i,j)(-1)^{i+j}$

b) If $a \in \mathbb{R}^n$ is such that $A\bar{a} = [-1, 1, -1, 1, \dots]$ then $\|A^{-1}\|_\infty = \|a\|_\infty$.

$$A^{-1} = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & & \vdots & \ddots \end{bmatrix} \text{ checkerboard}$$

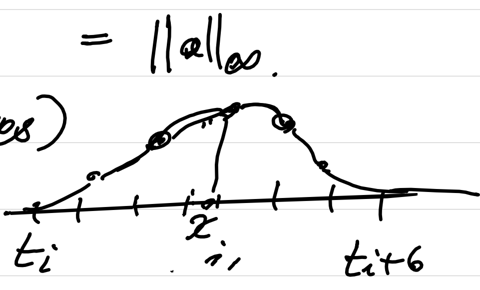
Proof: a) By Cramer's rule,
$$A^{-1}(i,j) = (-1)^{i+j} \frac{\overset{\ge 0}{\det A(\backslash j, \backslash i)}}{\underset{> 0}{\det A}} > 0.$$
remove row/column.

b) $\|A^{-1}\|_{\ell_\infty} = \max_{1 \le i \le n} \sum_{j=1}^n |A^{-1}(i,j)| = \max_{1 \le i \le n} \sum_{j=1}^n A^{-1}(i,j)(-1)^{i+j}$

$= \max_{1 \le i \le n} (-1)^i \sum_{j=1}^n A^{-1}(i,j)(-1)^j = \max_{1 \le i \le n} (-1)^i a_i$

(since $\|a\|_\infty \le \|A^{-1}\|_\infty$) $= \|a\|_\infty$.

Example: $k = 6$, (quintic splines)



$$A = \frac{1}{120} \begin{bmatrix} 66 & 26 & 1 & 0 \\ 26 & 66 & 26 & 1 \\ 1 & 26 & 66 & \ddots \\ 0 & \ddots & \ddots & \ddots \end{bmatrix} \Rightarrow \gamma = \frac{66 - 2 \cdot 26 - 2 \cdot 1}{120} = 12/120$$

$\|A^{-1}\| \le \gamma = 1/10.$

APPROXIMATION THEORY
LECTURE 12

LEMMA: Given Δ, (x_i), k, Sch-Wh., then $\|P_x\| \le \|a^x\|_\infty$ where $s = \sum_j a_j^x N_j$ is such that

$s_x(x_i) = [-1, 1, -1, \dots]$.

Optimal interpolation: (Given Δ)

1) Seeking for x^* s.t. $\|P_{x^*}\| = \min_x \|P_x\|$
2) But only have estimate $\|P_x\|_{L_\infty} \le \|a^x\|_{L_\infty}$. We seek instead x_* s.t.
$\|a^*\|_\infty = \min_{s_x} \|a^x\|$.

This search $\longrightarrow$ the Chebyshev spline.

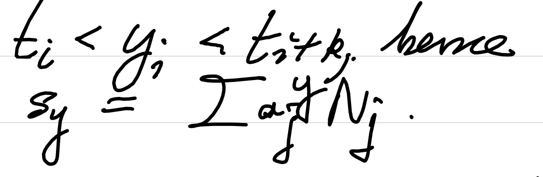
Def: The Chebyshev spline $T_* \in S_k(\Delta)$ is a spline that maximally (in times) equioscillates between ± 1.

$\left(\|f - l_\Delta\|_\infty \le \frac{1}{2^{n-1}} \cdot \frac{1}{n!} \|f^{(n+1)}\|_\infty \right)$

Theorem: The Cheb. spline T_* is the unique element in $S_k(\Delta)$ that minimizes $\|a^x\|$, and X_* are the points of its equioscillation.

The Remez algorithm:

1) Start with any x, let s_x be a spline s.t. $s_x(x_i) = (-1)^i$, $\|s_x\| > 1$ and $s_x = \sum a_j^x N_j$, then $(-1)^j a_j^x > 0$
2) This s_x has n local extrema (y_i).



One can show that Wh. cond are satisfied. $t_i < y_i < t_{i+k}$, hence $\exists s_y$ s.t. $s(y_i) = (-1)^i$; $s_y = \sum_j a_j^y N_j$.

$$(-1)^j a_j^x \ge (-1)^j a_j^y > 0.$$

3) In such a way, we obtain a sequence of $s_n = \sum_{j=1}^n a_j^m N_j$, $0 < (-1)^j a_j^{m+1} \le (-1)^j a_j^m$

then $\|s_m - s_*\|_\infty \to 0$ ($s_m \longrightarrow s_*$) $m \to \infty$. $(-1)^i s_x(x_i^*) = 1 = \|s_*\|$.
Given $x \longrightarrow x_*$ and s_* with desired properties, i.e. $\|a^*\|_{\ell_\infty} \le \|a^x\|_{\ell_\infty}$.

LEMMA: let $s_* = \sum_{j=1} a_j^* N_j$ satisfy $(-1)^i s(x_i) = 1 = \|s_*\|$.
Then $(-1) \cdot a_i^* = \|\lambda_i\|$, $\lambda_i(N_j) = \delta_{ij}$.
hence s_* (hence a^*) are uniquely determined.

Proof: Define the functional λ_i by the rule
$\lambda_i(s) = \sum A_{x^*}^{-1}(i,j) \cdot s(x_j^*)$, $A_x = (N_m(x_j))$.
Then $\lambda_i(N_m) = \underbrace{\sum A_{x^*}^{-1}(i,j)\, N_m(x_j^*)}_{A_x^{-1} \cdot A_x} = \delta_{im}$.

Further, $|\lambda_i(s)| \le \sum |A_x^{-1}(i,j)| \cdot \|s\|_\infty$.
$\Rightarrow \lambda_i(s_*) = \sum A_x^{-1}(i,j) \cdot s(x_j^*)$
$= \sum A_x^{-1}(i,j)(-1)^j$
$\Rightarrow \|\lambda_i\|_\infty = |\lambda_i(s_*)| = \sum |A_x^{-1}(i,j)|$.
$|a_i^*| = (-1)^i a_i$.

Thm (Demko): $\forall \Delta$, $\exists P_{x^*}$ with $\|P_{x^*}\| \le d_k$ independently of Δ.

Proof: Take x^* as Cheb. spline's equioscillations, then $\|P_{x^*}\|_{L_\infty} \le \|a^*\|_{L_\infty} = \max_i |a_i^*| = \max_i \|\lambda_i\| \le d_k$.

Thm (Scherer–Shadrin)

$\forall k$ $d_k := \sup_\Delta \max_i \|\lambda_i\|_\infty \le k \cdot 2^k$.

(B-spline basis condition number).
(also have $2^{k-3/2} < d_k < k \cdot 2^k$).

Approximation Theory
Lecture 13

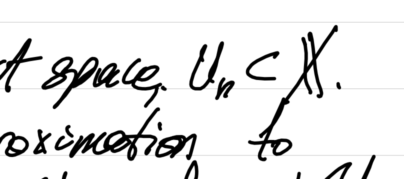
§13.1 Orthogonal spline projector

Def: $\mathbb{X}$ an inner product space i.e., we have a scalar (scalar product) s.t.
1) $(f,g)=(g,f)$ 2) $(f,f)\geq 0$ with equality iff $f=0$
3) (f,g) is linear.

With $\|f\|:=\sqrt{(f,f)}$, $\mathbb{X}$ becomes a linear normed space.

E.g.: 1) $x\in\mathbb{R}^n$, $\|x\|_2:=\sqrt{\sum_{i=1}^n |x_i|^2}$,
2) L_2: $(f,g)=\int_a^b f\cdot g\,dt \Rightarrow \|f\|_{L_2}=\int_a^b |f|^2 dt$

LEMMA: $\mathbb{X}$ an inner product space, $U_n\subset\mathbb{X}$. Then $u^*\in U_n$ is the best approximation to $f \iff (f-u^*,v)=0$ $\forall v\in U_n$, or $f-u^*\perp U_n$.

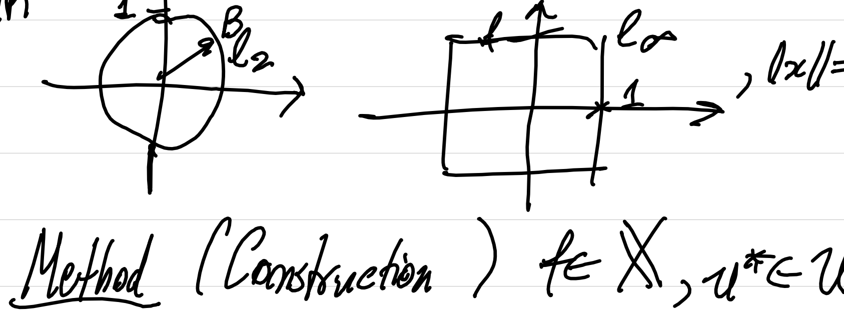


Proof: $\Leftarrow$ If $(*)$ is true, then $\forall u\in U_n$, $\|f-u\|^2=\|f-u^*+v\|^2$
$=\|f-u^*\|^2+\|v\|^2 \geq \|f-u^*\|^2 \Rightarrow u^*$ is a b.a.

Proof: $\Rightarrow$ If $\underset{\gamma}{(f-u^*,v)}\neq 0$ for some $v\in U_n$, then $\|f-u\|^2=\|(f-u^*)+\lambda v\|^2$
$=\|f-u^*\|^2+2\lambda\gamma+\lambda^2\|v\|^2<\|f-u^*\|^2-\varepsilon$
$u=u^*-\lambda v$

Remark: 1) U_n finite dimensional $\Rightarrow$ b.a. exists (for any norm).
2) $\|\cdot\|$ is strictly convex $\Rightarrow$ b.a. is unique (if it exists)
$\|f+g\|\leq\|f\|+\|g\|$ strict if $f\neq\lambda g$ $\hookrightarrow$ the unit sphere $\|f\|=1$ is strictly convex.

$\mathbb{R}^2$



, $\|x\|=\max(|x_1|,|x_2|)$.

Method (Construction) $f\in\mathbb{X}$, $u^*\in U_n$

$$u^*=\sum_{j=1}^n a_j u_j \quad (u_i)_{i=1}^n \text{ is a basis of } U_n.$$

take $v\in(u_i)_{i=1}^n$, $(u^*,u_i)=(f,u_i)$
$=\sum_{j=1}^n (u_i,u_j)a_j$

$\Rightarrow G\bar{a}=\bar{b}$ ← normal equation
$G=[(u_i,u_j)]_{i,j=1}^n$, $b=(f,u_i)$.
Gram matrix

Def: (Dual basis) Dual basis $(u_i^*)\in U_n$ if $(u_i^*,u_j)=\delta_{ij}$.

LEMMA: let $u_k^*=\sum_{j=1}^n b_{jk}u_j$
Then $G^{-1}=(b_{ij})$ $\qquad G^{-1}=\begin{array}{c} u_1^* \\ \begin{bmatrix} \begin{array}{c} b_{11}\\ b_{21}\\ \vdots\\ b_{n1}\end{array} & \cdots & \end{bmatrix}\end{array}$

Proof: $H=(b_{ij})$
$\Rightarrow (GH)_{ik}=\sum_{j=1}^n (u_i,u_j)b_{jk}$
$=\Big(u_i,\underbrace{\sum_{j=1}^n b_{jk}u_j}_{u_k^*}\Big)=\delta_{ik}.$

$\Rightarrow G^{-1}=H.$

§13.2 Orthogonal Spline Projector

Def: $P_{S_k(\Delta)}=P_S: C[a,b]\to S_k(\Delta)$,
with $(\underbrace{P_S f}_{\text{b.a. to } f \text{ from } S},v)=(f,v)$ $\forall v\in S_k(\Delta)$.

Remark: $\|P_S\|_2=1$ (In fact, for any orthogonal projector).

But we want $\|P_S\|_\infty=\dfrac{\|P_S f\|_\infty}{\|f\|_\infty}$.

Method: We changed the previous one a bit, so we want $P_S f=\sum_{j=1}^n a_j N_j$. $\sum N_j\equiv 1$

$$\sum_{j=1}^n (M_i,N_j)a_j=(P_S f,M_i)=(f,M_i)\Rightarrow$$
$\int M_i=1$

$G\bar{a}=\bar{b}$
$\hookrightarrow (M_i,N_j)$ $\qquad \sum_{j=1}^n g_{ij}=(M_i,\sum N_j)=1.$

LEMMA: $\|P_S\|_\infty\leq\|G_\Delta^{-1}\|_\infty\leq c_k$

Proof: The same as for interpolation.
$\|P_S f\|=\|\sum a_j N_j\|\leq\|\bar{a}\|_\infty$.
$\|a\|=\|G^{-1}b\|\leq\|G^{-1}\|_\infty\cdot\|b\|_\infty$
$|b_i|=|(f,M_i)|=\left|\int_a^b f(t)M_i(t)\,dt\right|\leq\|f\|_\infty\cdot\underbrace{\|M_i\|_1}_{1}$

$\Rightarrow \|b\|_\infty\leq\|f\|_\infty \Rightarrow \|a\|_\infty\leq\|G^{-1}\|_\infty\cdot\|f\|_\infty$.
$\Rightarrow \|P_S\|_\infty\leq\|G^{-1}\|_\infty$

Thm. If $M=\sup_{i,j}\dfrac{|y_i|}{|y_j|}<\infty$ where $y_i=[t_i,t_{i+k}]$.
then $\|P_{S_k(\Delta)}\|_\infty\leq M^{1/2}\cdot c_k$

Proof: Set $\hat{N}_i=\left|\dfrac{k}{y_i}\right|^{1/2}N_i=\left(\dfrac{y_i}{k}\right)^{1/2}\cdot M_i$.
(Then $\frac{1}{d_k}\|a\|_{\ell_2}\leq\|\sum a_i N_i\|_2\leq\|a\|_{\ell_2}$).
$\hat{G}=(\hat{N}_i,\hat{N}_j)$ Then one can show $\|\hat{G}\|_\infty<c_k$ independently of Δ.
Hence, $(M_i,N_j)=\left(\dfrac{k}{y_i}\right)^{1/2}(\hat{N}_i,\hat{N}_j)\left(\dfrac{k}{y_j}\right)^{-1/2}$
$\Rightarrow G=D^{-1/2}\hat{G}D^{1/2}\Rightarrow G^{-1}=D^{-1/2}\hat{G}^{-1}D^{1/2}$.
$\Rightarrow b_{ij}\leq\left(\dfrac{y_j}{y_i}\right)^{1/2}\hat{b}_{ij}\leq M^{1/2}\hat{b}_{ij}\Rightarrow\|G\|_\infty\leq M^{1/2}\cdot\|\hat{G}\|_\infty\leq M^{1/2}\cdot c_k$.

Thm: $\forall\Delta$ $\|P_S\|_\infty\leq c_k$

APPROXIMATION THEORY

LECTURE 14

§ 14. Wavelets $\longrightarrow$ construction of orthonormal bases

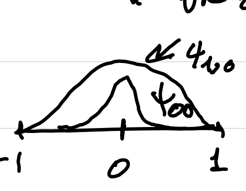
$f \in L_2(\mathbb{R})$: $\int_{\mathbb{R}} |f|^2 dx < \infty$

$(f, g) = \int_{\mathbb{R}} f \bar{g} \, dx$

Def: ψ is a wavelet if its translations (shifts) and dilations $\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)\}$ form a complete orthonormal system.

normalisation factor

Orthonormal: 1) $\psi_{jk} \perp \psi_{\ell m} \Leftrightarrow (\psi_{jk}, \psi_{\ell m}) = \delta_{j\ell}\delta_{km}$
2) $\|\psi_{jk}\|_2 = 1$.

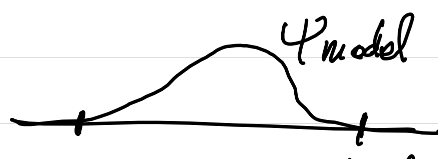
$\psi = \psi_{00} \Longrightarrow$ 

ψ_{jk} will be "centred" around $k/2^j$.

Complete ONS: means that $f = \sum_{j,k \in \mathbb{Z}} (f, \psi_{jk}) \psi_{jk}$.

L_2 sense

i.e. $\| f - \sum_{\substack{|j| \le M, \\ |k| \le N}} (\cdot) \psi_{jk} \|_{L^2} \longrightarrow 0$, $M \to \infty$, $N \to \infty$.

wavelet = small wave = a "bump" 

Similar to the Fourier expansion:

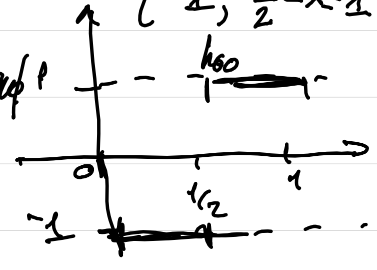
$f \in L_2(\mathbb{T}) \Rightarrow f = \sum_{k \in \mathbb{Z}} (f, e^{ikx}) e^{ikx}$

$\{e^{ikx}\}$ are dilations of the single function e^{ix}

Difference: e^{ix} hence e^{ikx} are not "locally" supported.

Example: The Haar wavelet $h(x) = \begin{cases} -1, & 0 \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} < x \le 1 \end{cases}$

Clear that $\{h_{jk}\}$ form an orthonormal system,



$h_{jk} \perp h_{jm}$ $m \ne k$

since supports are non-intersecting.

$h_{jk} \perp h_{\ell m} \Longrightarrow \int h_{jk} \cdot h_{\ell m} = 0$

Completeness $\Leftrightarrow$ $(f, \psi_{jk}) = 0$ $\forall j, k \Rightarrow f \equiv 0$.

Assumptions: Start with $\varphi = \varphi_{00}$, $\varphi_{0n} = \varphi(x-n)$ and assume that $\varphi_{0k} \perp \varphi_{0n}$ $\forall k, n$.

$\Leftrightarrow \varphi_{00} \perp \varphi_{0n}$ $\forall n$.

We set $V_0 = \overline{\text{span}(\varphi_{0n})_{n \in \mathbb{Z}}}$ in $L_2(\mathbb{R})$.

$f = \sum_{n \in \mathbb{Z}} c_n \varphi_{0n}$, then $f \in L_2 \Leftrightarrow \sum c_n^2 < \infty$.

$V_j = \overline{\text{span}(\varphi_{jn})_{n \in \mathbb{Z}}}$ in $L_2(\mathbb{R})$

$f \in V_j \Leftrightarrow f = \sum_n c_n \varphi_{jn}$, $\sum c_n^2 < \infty$

Assumption 2: $V_0 \subset V_1 \Rightarrow \varphi(x) = \frac{1}{\sqrt{2}} \sum_k a_k \varphi_{1k}(x)$
$= \sum_k a_k \varphi(2x - k)$.

Def: $\varphi(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k)$ $(*)$

scaling function — refinement equation.

Example: characteristic function $\chi_{[0,1]} = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

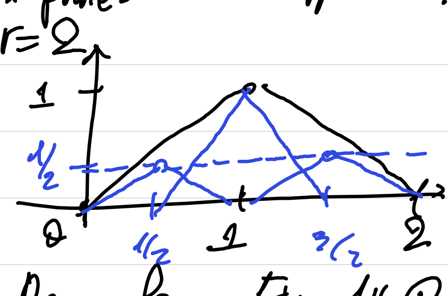
Then $\chi_{[0,1]} = \chi_{[0,1/2]} + \chi_{[1/2,1]} = \chi(2x) + \chi(2x-1)$.

Example: B-spline on the integer knots:

$N(x) = N_{0,r}(x) = 2^{-r+1} \sum_{k=0}^{r} \binom{r}{k} N(2x - k)$

support $[0, r]$ of order r.

lin. splines $r = 2$



$N_0 = \frac{1}{2}\hat{N}_0 + \hat{N}_1 + \frac{1}{2}\hat{N}_2$
$= \frac{1}{2}(\hat{N}_0 + 2\hat{N}_1 + \hat{N}_2)$

Remark: Ex. 14.2 Don't start until further notice.

Shifts of B-spline are NOT orthogonal.

Construction: let $\varphi \perp \varphi(\cdot - n)$ and φ satisfies $(*)$.

Then $\psi(x) = \sum_{k \in \mathbb{Z}} (-1)^k a_{1-k} \varphi(2x - k)$ is an orthonormal wavelet.

Example: Take $\chi(x) = \underset{a_0}{\chi(2x)} + \underset{a_1}{\chi(2x-1)}$.

$\Rightarrow h(x) = \chi(2x) - \chi(2x-1)$ is a wavelet.

$\varphi_{00}(x) = \sum_{k \in \mathbb{Z}} a_k \cdot \varphi_{0k}(2x)$

$\Downarrow$

$\varphi_{0n}(x) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} a_k \cdot \varphi_{1, k+2n}(x)$

$\Downarrow$

$\psi_{0n}(x) = \frac{1}{\sqrt{2}} \sum_k (-1)^k a_{1-k} \varphi_{1, k+2n}$ $(***)$

LEMMA: $\psi_{0n} \perp \varphi_{0m}$, $\psi_{0n} \perp \psi_{0m}$ $\forall m, n$.

(Sufficient to prove $m = 0$).

Proof: $(\psi_{00}, \psi_{0n}) = \frac{1}{2} \sum_k (-1)^k a_{1-k} (-1)^{k-2n} a_{1-k+2n}$

$= \frac{1}{2} \sum_k a_{1-k} a_{1-k+2n} = \frac{1}{2} \sum_k a_k \cdot a_{k+2n}$

$= (\varphi_{00}, \varphi_{0n}) = \delta_{0n}$.

Approximation Theory

Lecture 15

§ 15. Wavelets

Lemma Let $\varphi \in L_2(\mathbb{R})$ satisfy

1) $\varphi(x) = \sum_{k\in\mathbb{Z}} a_k \varphi(2x-k)$

2) $\varphi(x-k) \perp \varphi(x)$ $\forall n$.

Then, if we define
$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k a_{1-k}\, \varphi(2x-k)$$
we have

1) $\psi(x) \perp \psi(x-k)$
2) $\psi(x) \perp \varphi(x-k)$.

$$\varphi_{00} = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} a_k \cdot \varphi_{1k}, \quad \varphi_{0n} = \frac{1}{\sqrt{2}} \cdot \sum a_k \varphi_{1,k+2n}$$
$$= \frac{1}{\sqrt{2}} \cdot \sum_{k\in\mathbb{Z}} a_{k-2n} \cdot \varphi_{1k}$$

$$\delta_{0n} = (\varphi_{00}, \varphi_{0n}) = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k \cdot a_{k-2n}$$

Lemma: We have
$$\varphi_{1k} = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} a_{k-2n} \cdot \varphi_{0n} + (-1)^k a_{1-k+2n} \psi_{0n}$$

Lemma: We have $\underbrace{V_1}_{\overline{\operatorname{span}\{\varphi_{1n}\}}} = \underbrace{V_0}_{\overline{\operatorname{span}\{\varphi_{0n}\}}} \oplus \underbrace{W_0}_{\overline{\operatorname{span}\{\psi_{0n}\}}}$

Proof: from definition, $V_0 \subset V_1$, $W_0 \subset V_1$
$\Rightarrow V_0 + W_0 \subset V_1$.
And from lemma, $V_1 \subset V_0 + W_0$.

Need: $\overline{V_0 + W_0} = \overline{V_0} + \overline{W_0}$
by orthogonality.

Remark: $V_j = V_{j-1} \oplus W_{j-1}$

$V_0 = \overline{\operatorname{span} \varphi_{0n}}$
$V_1 = \overline{\operatorname{span}\{\varphi_{1n}\}}$
$= \overline{\operatorname{span}\{\varphi(2x-n)\}}$

Respectively,
$\cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots L_2(\mathbb{R})$

$V_j = \overline{\operatorname{span}\{\varphi_{jn}\}}$
$= \overline{\varphi(2^j x - n)}$

$W_j = \overline{\operatorname{span}\{\psi_{jn}\}}$

Lemma: Let $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = 0$, then ψ is an orthonormal wavelet, i.e. $\bigoplus_{j\in\mathbb{Z}} W_j = L_2(\mathbb{R})$ i.e. dilations and shifts of ψ form an ONB for $L_2(\mathbb{R})$.

Proof: NTS: $f \perp W_j\ \forall j \Rightarrow f \equiv 0$.

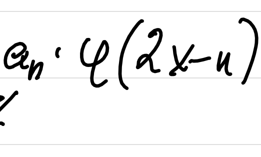
Let $P_j f$ be orthogonal projection on V_j, and Q_j orth. projection onto W_j. We have $Q_j(f) = 0\ \forall j$. On the other hand
$V_j = V_{j-1} + W_{j-1} \Rightarrow P_j(f) = P_{j-1}(f) + \underbrace{Q_{j-1}(f)}_{=0}$

Hence $P_j(f) = P_{j-1}(f)\ \forall j$. But $\overline{\bigcup V_j} = L_2(\mathbb{R})$
So $P_j(f) \to f \Rightarrow P_j(f) = f\ \forall j$. Finally,
$P_j f \in V_j$ & $\bigcap V_j = 0 \Rightarrow f = 0$.

Def: A multiresolution analysis (MRA), with generator φ, is a sequence of closed subspaces V_j s.t.

a) $V_{j-1} \subset V_j$
b) $f \in V_0 \Rightarrow f(2^j \cdot) \in V_j$
c) $\exists \varphi$ s.t. $\{\varphi(x-k)\}$ form an orthonormal basis of V_0.
d) $\overline{\bigcup V_j} = L_2(\mathbb{R})$, $\bigcap_{j\in\mathbb{Z}} V_j = 0$.

Theorem: Given a multiresolution analysis with generator φ, the function ψ defined in (*) is an orthonormal wavelet.

Thm: The Haar f-n h  is a wavelet.

§ 15.2 Construction

Need:

1) $\varphi(x) = \sum_{n\in\mathbb{Z}} a_n \cdot \varphi(2x-n)$

2) $\{\varphi(x-n)\}$ are orthonormal.

Lemma: We have $\{\varphi(x-k)\}$ are orthonormal
$\Leftrightarrow \sum_{k\in\mathbb{Z}} |\hat{\varphi}(x+2\pi k)|^2 \equiv 1$ a.e.

Proof: Apply Parseval's identity for the FT

$$\delta_{0n} = \int_{\mathbb{R}} \varphi(x)\overline{\varphi(x-n)}\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}\, \overline{\widehat{\varphi(\cdot-n)}}\,dt$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi} \cdot \overline{\hat{\varphi}} e^{int}\,dt$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{\varphi}|^2 e^{int}\,dt$$
$$= \frac{1}{2\pi}\sum \int_{2\pi k}^{2\pi(k+1)} = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{int}\,dt \Leftrightarrow F(t) \equiv 1.$$

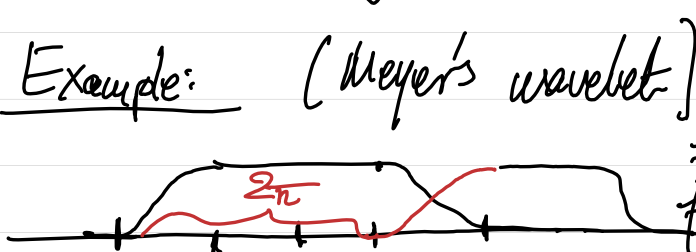
Apply FT to (1). We get
$$\hat{\varphi}(t) = \underbrace{m(t/2)}\cdot \hat{\varphi}(t/2) \quad m(t) = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n e^{-int}$$

Thm: Let f satisfy the following:

1) $f(2t) = m(t) \cdot f(t)$, $m(t)$ 2π-periodic $\in L_2(\mathbb{T})$
2) $\sum |f(x+2\pi k)|^2 \equiv 1$

Then $f^{\vee}$ is a generator for MRA.

Example: (Meyer's wavelet)



$f \equiv 1$ on $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$
$f \equiv 0$ outside $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$.

1) $f(2t) = m(t) \cdot \overset{\equiv 1}{f(t)}$
$\operatorname{supp} = [-2\pi/3, 2\pi/3]$

2) $\sum |f(t+2\pi k)|^2 \equiv 1$ since $f_1^2 + f_2^2 = 1$ by def.
For each t, $\sum$ has 2 terms at most.

Example: (Shannon's Wavelet).
$f \equiv 1$ on $[-\pi, \pi]$
$\equiv 0$ elsewhere.

$$\sum |f(x+2\pi k)|^2 = 1, \quad f^{\vee} = \frac{\sin \pi x}{\pi x} = \varphi$$

V_0 = all functions g s.t. $\operatorname{supp}\hat{g} \subset [-\pi, \pi]$.
$$g(x) = \sum_n g(n)\varphi(x-n).$$

$$\varphi(x) = \sum \varphi(n/2)\varphi(2x-n)$$

$$\psi(x) = 2\varphi(2x-1) - \varphi(x-1/2)$$

APPROXIMATION THEORY
LECTURE 16

Remark: If shifts are not orthogonal (like for the B-spline).

$$\hat{\Phi} = \frac{\hat{\varphi}(t)}{F_\varphi(t)^{1/2}}, \quad F_\varphi(t) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(t+2\pi k)|^2$$

$\Rightarrow F_\Phi(t) = 1. \quad F(t) > \gamma > 0.$

$F(t) > 0$ from stability:

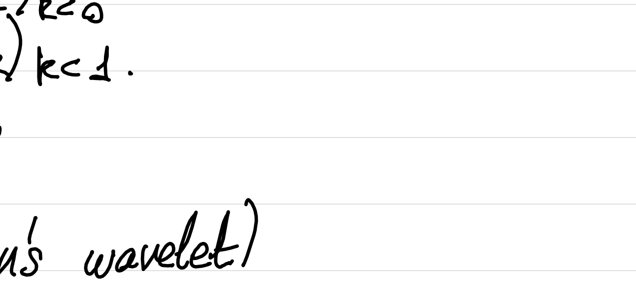
$$\gamma_1 \cdot \|c\|_{\ell_2} \le \left\| \sum_n c_n \varphi_{0n} \right\|_{L_2} \le \gamma_2 \cdot \|c\|_{\ell_2} \quad \forall c \in \ell_2.$$

if orthonormal, then have equality.

$\Rightarrow \gamma_1^2 \le F(t) \le \gamma_2^2.$

Also, $\overline{\operatorname{span}\{\Phi_{0n}\}} = V_0 = \overline{\operatorname{span}\{\varphi_{0n}\}}$

Orthonormalisation of B-spline sequence:



$\varphi_{00} = N_0$ - b.a. from $(N_k)_{k<0}$. (L_2 sense).

$\Rightarrow \varphi_{00} \perp (N_k)_{k<0}$

$\varphi_{01} \perp (N_k)_{k<1}.$

$\perp \varphi_{00}$

Example (Shannon's wavelet)

$$f(t) = \begin{cases} 1, & t \in [-\pi, \pi] \\ 0, & \text{otherwise.} \end{cases}$$

$f(2t) = m(t) \cdot f(t), \quad m(t) = \frac{1}{2} \sum_{n \in \mathbb{N}} a_n e^{-int}$

$f(2t) = \chi_{[-\pi/2, \pi/2]}$

$$m(t) = \chi_{[-\pi/2, \pi/2]}, \quad a_k = \frac{2}{2\pi} \int_{-\pi/2}^{\pi/2} e^{int} \, dt = \frac{\sin \pi k/2}{\pi k/2}.$$

Two properties: 2) $\{\varphi_{0n}\}$ are orthonormal $\Leftrightarrow$ $F(t) = \sum |\hat{\varphi}(t + 2\pi k)|^2 = 1.$

1) refinement eq-n $\Leftrightarrow \hat{\varphi}(t) = m(t/2) \cdot \hat{\varphi}(t/2).$

Goal: Find φ based on $m(t) = \frac{1}{2} \sum a_n \cdot e^{-int}.$

LEMMA: Under 1)-2), we have

$$|m(t)|^2 + |m(t+\pi)|^2 = 1$$

Proof: $(1 =) F(2t) = \sum |\hat{\varphi}(2t + 2\pi k)|^2$

$$= \sum_{k\in\mathbb{Z}} |m(t+\pi k)| \cdot |\hat{\varphi}(t+\pi k)|$$

$$= \sum_{k \text{ odd}} + \sum_{k \text{ even}}$$

$$= \sum |m(t+2\pi k)|^2 \cdot |\hat{\varphi}(t+2\pi k)|^2 + \sum |\underbrace{m(t+\pi+2\pi k)}_{m(t+\pi)}|^2 \cdot |\hat{\varphi}(t+\pi+2\pi k)|^2$$

$$= |m(t)|^2 F(t) + |m(t+\pi)|^2 \cdot F(t+\pi) = F(2t).$$

Remark Necessary, but not sufficient (Ex 16.1).

Next, suff. cond. (but not necessary).

LEMMA: Let φ satisfy the refinement eq-n, $\hat{\varphi}(0) = 1$, and $\hat{\varphi}(t) = m(t/2)\hat{\varphi}(t/2)$ $(**)$ and

1) $|m(t)|^2 + |m(t+\pi)|^2 = 1,$

2) $|m(t)| > 0$ on $[-\pi/2, \pi/2]$

3) $m(0) = 1.$

Then $\{\varphi_{0n}\}$ are orthonormal $\Leftrightarrow F(t) = 1.$

Proof:

Will prove $F(0) = 1$ and $F(t) \equiv \text{const}$

1) Have $m(0) = 1 \Rightarrow m(\pi) = 0 \Rightarrow m((2l+1)\pi) = 0.$

$t = 2(2l+1)\pi \overset{(**)}{\Rightarrow} \hat{\varphi}(t) = 0$

$t = 2^j(2l+1)\pi \Rightarrow \hat{\varphi}(t) = 0.$

So for all $t = 2\pi k$, $\hat{\varphi}(t) = 0 \Rightarrow F(0) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(2\pi k)|^2 = |\varphi(0)|^2 = 1$

2) $F(t) = |m(t/2)|^2 \cdot F(t/2) + |m(t/2+\pi)|^2 \cdot F(t/2+\pi)$

Let $\min_{t\in[-\pi,\pi]} F(t) = F(t_0) = \gamma$, $t_1 = t_0/2 \in [-\pi/2, \pi/2].$

$$\overset{\gamma}{\overset{\parallel}{F(t_0)}} = \underbrace{|m(t_1)|^2}_{>0} \underset{\underset{\gamma}{\parallel}}{F(t_1)} + |m(t_1+\pi)|^2 \cdot \underset{\underset{\gamma}{\parallel}}{F(t_1+\pi)}.$$

$$\left.\begin{array}{l} \text{Repeat } t_j = \dfrac{t_0}{2^j} \Rightarrow F(0) = \gamma = \min F(t) \\ \min \to \max \Rightarrow F(0) = \gamma = \max F(t) \end{array}\right\} = \text{const}$$

Q: How to get φ (with refinement eqn) from $m(\cdot)$?

We have $\hat{\varphi}(t) = \hat{\varphi}(t/2) \cdot m(t/2) = \hat{\varphi}(t/4) m(t/4) m(t/2) = \ldots$

Define $m_k(t) = \prod_{j=1}^{k} m(t/2^j)$

then $\hat{\varphi}(t) = \lim_{k\to\infty} m_k(t)$ $(***)$ (if the limit exists).

Thm (Daubechies) Let $m(t) = \frac{1}{2} \sum_{k=0}^{N} a_k e^{-ikt}$ be a trig. pol. that satisfies $|m(t)|^2 + |m(t+\pi)|^2 = 1.$ Then the limit $(***)$ exists and defines a continuous function φ with finite support $[0, N]$ which is a generator for MRA.

Construction of such $m(t)$. Start with $M(t) = \sum_{k=0}^{N} b_k \cos kt \ge 0.$

$$1 = (\cos^2 t/2 + \sin^2 t/2)^{2K+1}$$

$$= \Big(\underbrace{\sum_{k=0}^{K}}_{M(t)} + \underbrace{\sum_{k=K+1}^{2K+1}}_{M(t+\pi)} \Big) \binom{2K+1}{k} (\cos^2 t/2)^{2K+1-k} (\sin^2 t/2)^k$$

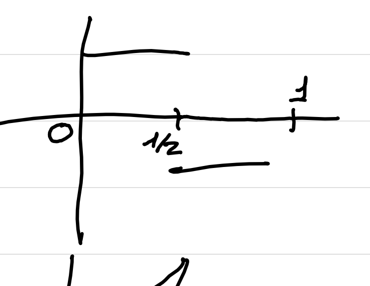
$M(t) = |m(t)|^2$ Then, by Fejér-Riesz thm

$\exists\, m(t) = \frac{1}{2} \sum_{k=0}^{N} a_k e^{-ikt}$ s.t. $|m(t)|^2 = M(t).$

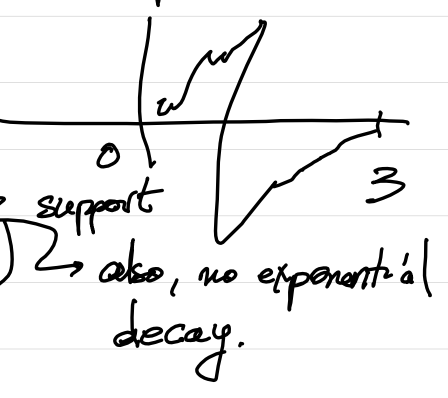
If $M > 0$ (cosine poly).

Smoothness $\exists c_0$ s.t. $\forall r \; \exists \psi$ s.t. $\psi \in C^r$, $|\operatorname{supp} \psi| \le c_0 \cdot r.$

$N = 1, k = 0 \Rightarrow$ Haar wavelet



$N = 3, k = 1 \Rightarrow$ Daub. wavelet $\in \operatorname{Lip} \alpha, \alpha = 0.55$



Remark: Analytic wavelet $\Rightarrow$ infinite support (like Shannon's wavelet) $\to$ also, no exponential decay.

$$\varphi = \frac{\sin \pi x}{\pi x}$$